Quantum Mechanics GRE Notes

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Here is some stuff that you should know about quantum mechanics for the GRE.

- **Schrödinger Equation.** General and time-independent:

  \[ i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V \right) \Psi \]

  \[ H\psi = E\psi \quad \Psi = \psi e^{-iEt/\hbar} \quad (1) \]

- **Formalism and Operator Algebra.** This is probably actually the easiest topic for people who have taken 143a. Some equations of use are:

  \[ \langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^* \]

  \[ \hat{O} = \hat{O}^\dagger \text{ if } \hat{O} \text{ is Hermitian} \]

  \[ \langle \alpha | \hat{O} | \beta \rangle = \beta \langle \alpha | \beta \rangle \text{ if } |\beta\rangle \text{ is an eigenstate of } \hat{O} \]

  \[ = \int_{-\infty}^{\infty} dx \alpha^*(x)\hat{O}(x)\beta(x) \text{ in 1D} \]

  \[ = A^\dagger \times O \times B \text{ as matrices} \]

  \[ [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \]

  \[ . \quad [\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}] \]

  \[ [\hat{A}\hat{B}, \hat{C}] = [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}] \quad (2) \]

  A Hermitian matrix has real eigenvalues; observables thus must be represented by Hermitian operators.

- **Position and momentum.**

  \[ \hat{p} = -i\hbar \nabla \]

  \[ [\hat{x}, \hat{p}] = i\hbar \]

  \[ [f(x), \hat{p}] = i\hbar \frac{df}{dx} \quad (3) \]
• **De Broglie equation.** This one always trips me up even though it’s easy:

\[
\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mkBT}} \tag{4}
\]

if the particle’s energy is thermal. Fun fact: the De Broglie wavelength of thermal neutrons is about 1.78 angstroms.

• **Emitted photons.** Yeah. The name of the game:

\[
E = -\mu e^2 Z^2 \frac{n^2}{n^2} = (-13.6 \text{eV}) \frac{Z^2}{n^2} \text{ for } \mu = m_e 
\]

\[
\therefore \lambda^{-1} \propto Z^2 \left( \frac{1}{n^2} - \frac{1}{n^2} \right) \tag{5}
\]

This means the key to their beloved positronium problems is that \(\mu \rightarrow \mu/2\) since the reduced mass is now \((m_e^{-1} + m_e^{-1})^{-1}\). So basically you can do the problem as if it were hydrogen with 13.6 eV \(\rightarrow\) 6.8 eV.

• **Hydrogen Ground State.** A good thing to know:

\[
\psi(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \tag{6}
\]

• **Infinite Square Well.** Straightforward:

\[
\psi_n = \frac{1}{\sqrt{a}} \sin \frac{n\pi x}{a} \\
E_n = \frac{\pi^2n^2\hbar^2}{2ma^2} \tag{7}
\]

Pay attention to how \(E_n\) scales with \(n\). Also, \(n \geq 1\) of course.

• **Singlet and triplet states.** These are useful from two perspectives. The first point of view is taking them as eigenstates of particle exchange in a two-particle system. The second is ways of adding two spin-\(\frac{1}{2}\) particles to get one spin-1 particle. From the first perspective, the singlet state has eigenvalue -1; from the second, it has \(s = m = 0\):

\[
|0 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{8}
\]

The triplet states have eigenvalue 1 with regards to parity and have \(s = 1\) and \(m \in \{-1, 0, 1\}:

\[
|1 1\rangle = |\uparrow\uparrow\rangle \\
|1 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
|1 -1\rangle = |\downarrow\downarrow\rangle \tag{9}
\]
• **Selection rules.** They seem to like making you remember these.

\[
\begin{align*}
\Delta l & = \pm 1 \\
\Delta m_l & = \pm 1 \; \text{or} \; 0 \\
\Delta j & = \pm 1 \; \text{or} \; 0
\end{align*}
\]  

(10)

Here \( l \) is the orbital angular momentum; \( m_l \) is the \( z \) component of \( l \); \( j = l + s \) is the total angular momentum. (I’m not exactly sure how you can get \( \Delta j = 0 \) but I read it on the internet so it must be true!) Typically \( \Delta n = \pm 1 \) but there is no reason that this must hold. Just remember that the emitted photon is spin 1 and so its angular momentum can be described as one of \([0 \; 0], [1 \; 1], [1 \; 0], \) or \([1 \; -1] \). The phrase “electric dipole radiation” implies that \( \Delta l = 0 \) is not physically permissible, even though it is allowed mathematically. Electric dipole transitions are “allowed”, while magnetic dipole or electric quadrupole transitions are “forbidden” (but do actually occur).

• **Observed states and expectation values.** The usual format of such a problem is

\[
|\psi\rangle = \sqrt{p_1}|k_1\rangle + \sqrt{p_2}|k_2\rangle + \sqrt{p_3}|k_3\rangle + \cdots
\]

(11)

The expectation value is

\[
p_1 k_1 + p_2 k_2 + p_3 k_3 + \cdots
\]

and the probability of being in a state \(|k_i\rangle\) is just \( p_i \). This is all assuming the \(|k_i\rangle\) are eigenkets of whatever operator you’re finding the expectation value of.

• **Heisenberg relation.** The general version:

\[
\Delta A \Delta B \geq \frac{1}{2} |[A, B]| 
\]

(12)

In particular,

\[
\Delta x \Delta p \geq \frac{\hbar}{2} \\
\Delta E \Delta t \geq \frac{\hbar}{2}
\]

(13)

• **Time Independent First-Order Perturbation Theory.** Given a perturbation \( H' \) to some original Hamiltonian with eigenstates \(|\psi_n^0\rangle\) and energies \( E_n^0 \), the perturbed energies and states are:

\[
\begin{align*}
E_n^1 & = E_n^0 + \langle \psi_n^0 | H' | \psi_n^0 \rangle \\
\psi_n^1 & = \psi_n^0 + \sum_{m \neq n} \frac{\langle \psi_m^0 | H' | \psi_n^0 \rangle}{E_n^0 - E_m^0} \psi_m^0
\end{align*}
\]

(14)

Recall that

\[
\langle \psi_n | \hat{O} | \psi_m \rangle = \int_{-\infty}^{\infty} \psi_n^*(x) \hat{O}(x) \psi_m(x) dx
\]

in one dimension. (You will often be perturbing the infinite square well.)

• **Fundamental Understanding of the Concepts of Quantum Mechanics.** Yeah. Whatever.